THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

AREAL COORDINATES.

THE advantage of Areal "over Cartesian coordinates is in the greater symmetry of the analytical forms, and in the more convenient treatment of the line infinity, and of points at infinity."—Cayley, Art. "Geometry," *Encyc. Britt.*, 9th Edn.

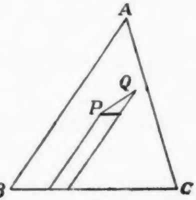
So many questions either directly on Areals,* or easily and elegantly treated by means of that system of coordinates, have appeared in recent years in the various Entrance Scholarship Examinations that no apology is necessary for introducing such questions into the following paper, which contains solutions of all those set at Cambridge from 1894-1900.

The usual formulae will be directly established without recourse to Cartesians. As in Geometry of Three Dimensions, there is no reason why at this stage of a student's work the notation of the Calculus should not be used for the sake of brevity and for the treatment of maxima and minima.

Space does not permit proofs of such propositions as "The general equations of the first and second degree represent respectively a straight line and a conic."

SECTION I.

1. DEFINITION. If M be any point in the plane of the triangle of reference ABC, and if the ratios $\dfrac{\triangle BMC}{\triangle ABC}, \dfrac{\triangle CMA}{\triangle ABC}, \dfrac{\triangle AMB}{\triangle ABC}$ be denoted by x, y, z respectively, then x, y, z are the areal coordinates of M.



[If the areas BMC, CMA, AMB are taken as the coordinates of M, we have the system of Barycentric Coordinates of Möbius.]

The relation between the coordinates of a point is clearly $\Sigma x = 1$.

2. *To find the distance ρ between two fixed points* $(x_1, y_1, z_1), (x_2, y_2, z_2)$.

From Fig.,
$$\rho^2 = (a\overline{z_2 - z_1})^2 + (c\overline{x_2 - x_1})^2 - 2(a\overline{z_2 - z_1})(c\overline{x_2 - x_1})\cos(180^\circ - B)$$
$$= -\Sigma a^2 (y_1 - y_2)(z_1 - z_2),$$
or
$$= \Sigma \overline{x_1 - x_2}^2 bc \cos A.$$

* Called *triangular* coordinates by Ferrers and Whitworth.

3. *To express the coordinates of a point on a straight line in terms of a single parameter.*

If (x_0, y_0, z_0) be a fixed point on the line $\Sigma lx=0$, and ρ the distance between (x_0, y_0, z_0) and any other point (x, y, z) on this line, then $\Sigma l(x-x_0)=0$ and $\Sigma(x-x_0)\equiv\Sigma x_0-\Sigma x=0$;

$$\therefore\ \frac{x-x_0}{m-n}=\ldots=\ldots=\lambda \text{ (say)},$$

and $\rho^2=-\lambda^2\Sigma a^2(n-l)(l-m)=\lambda^2[\Sigma a^2l^2-2\Sigma bc\cos A\,.\,mn]=\lambda^2t^2$ (say).

Then
$$\frac{x-x_0}{m-n}=\ldots=\ldots=\frac{\rho}{t}.$$

[Note $t^2=-\Sigma a^2(n-l)(l-m)=\Sigma(m-n)^2bc\cos A=\Sigma a^2l^2-2\Sigma bc\cos A\,.\,mn.$]

4. *To find the angle (θ) between two lines $\Sigma lx=0$, $\Sigma l'x=0$.*

Let the lines intersect in $O\,(x_0, y_0, z_0)$. Take P, P' one on each of these lines so that $OP=OP'=\rho$. The coordinates of P are

$$x_0+(m-n)\frac{\rho}{t},\ y_0+(n-l)\frac{\rho}{t},\ z_0+(l-m)\frac{\rho}{t}.$$

Similarly for P'. Then we have

$$PP'^2=\rho^2\Sigma\left(\frac{m-n}{t}-\frac{m'-n'}{t'}\right)^2bc\cos A=2\rho^2\left[1-\frac{1}{tt'}\Sigma(m-n)(m'-n')bc\cos A\right]$$

$$=2\rho^2\left[1-\frac{1}{tt'}(\Sigma a^2ll'-\Sigma(mn'+m'n)bc\cos A)\right].\quad \text{But } \cos\theta=1-\frac{PP'^2}{2\rho^2};$$

$\therefore\ \cos\theta=[\Sigma a^2ll'-\Sigma(mn'+m'n)bc\cos A]/t\,.\,t'$,

and after some reduction

$$\tan\theta=\pm2\Delta\Sigma(mn'-m'n)/[\Sigma a^2ll'-\Sigma(mn'+m'n)bc\cos A].$$

5. *Condition that two lines are perpendicular.*

By (4), $\Sigma a^2ll'=\Sigma(mn'+m'n)bc\cos A$,

or $\Sigma\left(l'\frac{\partial}{\partial l}\right)t^2=0.$

6. *Condition that two lines are parallel.*

By (4), $\Sigma mn'=\Sigma m'n.$

7. *The equation to the perpendicular from a given point to a given line.*

Let (f, g, h) be the given point, and (x, y, z) any point on the given line $\Sigma lx=0$.

Then ρ the distance between (f, g, h) and (x, y, z) is given by

$$\rho^2=\Sigma bc\cos A\,(x-f)^2.$$

Then for maximum or minimum values we have

$\Sigma bc\cos A\,(x-f)dx=0$ (1); subject to $\Sigma l dx=0$ (2); and $\Sigma dx=0$ (3).

Hence we have

$$\begin{vmatrix}(x-f)bc\cos A & \ldots & \ldots\\ l & \ldots & \ldots\\ 1 & \ldots & \ldots\end{vmatrix}\equiv\Sigma(x-f)(m-n)bc\cos A=0.$$

8. *The length of the perpendicular from a given point to a given straight line.*

By undetermined multipliers:

$(x-f)bc\cos A+Pl+Q=0$ (1), and two similar equations (2), (3);

$(1)\times(x-f)+(2)\times(y-g)+(3)\times(z-h)$ gives $\rho^2 = P\Sigma lf$; $\because\ \Sigma lx=0$;

$(1)\times a\sec A+(2)\times b\sec B+(3)\times c\sec C$ gives $P\Sigma la\sec A+Q\Sigma a\sec A=0$;

$(1)\times la\sec A+(2)\times mb\sec B+(3)\times nc\sec C$ gives

$$P\Sigma l^2 a\sec A+Q\Sigma la\sec A=abc\Sigma lf;$$

$$\therefore\ P\,.\,t^2=abc\,.\,\frac{abc}{4R^2}\Sigma lf=4\Delta^2\Sigma lf;$$

$$\therefore\ \text{length of perpendicular is } (2\Delta\Sigma lf)/t.$$

[Note that l, m, n are proportional to the perpendiculars on the line $\Sigma lx=0$ from the vertices of the triangle of reference.]

9. *The Line at Infinity.*

$(2\Delta\Sigma lf)/t$ is finite for all finite values of the quantities involved, except when $t=0$. In this case the length of the perpendicular from *any* point in the finite plane to the given line is infinite. Hence the line is out of the finite plane, *i.e.* is at infinity. Now if $t=0$, and l, m, n are real, we must have $l=m=n=\lambda$ (say), and the equation to the line becomes $\lambda\Sigma x=0$. Hence if x, y, z be proportional to the coordinates of any point on the line at infinity, then $\Sigma x=0$; thus $\Sigma x=0$ is called the equation to the line at infinity.

10. *The area of a triangle.*

Let $(x_1, y_1, z_1)\ldots$ be the coordinates of the vertices P, Q, R of the triangle PQR, area Δ'.

The perpendicular from P on QR is

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}\times(\text{some function of the coordinates of } Q \text{ and } R).$$

Hence

$$\Delta'=\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}\times(\text{some function independent of the coordinates of } P),$$

and $\therefore$ by symmetry, $\Delta'=\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}\times(\text{some constant}).$

To determine the constant let P, Q, R coincide respectively with A, B, and C.

Then $x_1=y_2=z_3=1$; $x_2=x_3=y_1=y_3=z_1=z_2=0$; and $\Delta'=\Delta$.

Hence the constant is Δ, and we have

$$\Delta'=\Delta\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}.$$

COROLLARY. The perpendicular from (x_0, y_0, z_0) on the line $\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix}$ is of length $\dfrac{2\Delta}{t}\begin{vmatrix} x_0, & y_0, & z_0 \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix}$. If ρ_{12} be the join of the two fixed points (x_1, y_1, z_1), (x_2, y_2, z_2), then from the above $t=\rho_{12}$.

11. *Transformation of coordinates.*

It is easily seen that changing the base triangle cannot alter the degree of an equation, so that the formulae of transformation are simple homographic, $x=\Sigma a_1 x'$, etc. But at A', $x'=1$, $y'=z'=0$. So $x_A=a_1$, $y_A=a_2$, $z_A=a_3$, *i.e.* A' is (a_1, a_2, a_3) referred to the original base triangle. Hence the scheme

$$\begin{array}{r|c|}
 & A' \quad B' \quad C' \\
\hline
x= & a_1x'+b_1y'+c_1z' \\
y= & a_2x'+b_2y'+c_2z' \\
z= & a_3x'+b_3y'+c_3z' \\
\hline
 & x' \quad y' \quad z'
\end{array}.$$

12. *Invariant of a straight line.*

If by transformation the equation of the line $\Sigma lx=0$ becomes $\Sigma l'x=0$, and the point (f, g, h) becomes (f', g', h'), then the perpendicular from the point to the line $=(\Sigma lf)/t=(\Sigma l'f')/t'$.

But $\Sigma lf\equiv\Sigma l'f'$ if no multiplier be introduced, and $\therefore\ t=t'$. Hence t is an invariant.

Again $\Sigma lx=0$ and $\Sigma lx=k$ represent any two parallel lines. The distance between them = the difference between the perpendiculars on them from any point $=(\Sigma lx)/t-(\Sigma lx-k)/t'=$ const.; $\therefore\ t=t'$.

13. *The general equation of the second degree.*

This represents two straight lines if

$$\begin{vmatrix} \phi_{xx}, & \phi_{xy}, & \phi_{xz} \\ \phi_{yx}, & \phi_{yy}, & \phi_{yz} \\ \phi_{zx}, & \phi_{zy}, & \phi_{zz} \end{vmatrix}=0, \text{ i.e. if } \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}=0.$$

The lines are perpendicular if $\Sigma a^2u=2\Sigma bcu'\cos A$.

If they are parallel the result of eliminating z from $\phi(x, y, z)=0$, and $\Sigma x=0$ must be a quadratic function of x and y, which is a perfect square. Whence we have

$$\Sigma U+2\Sigma U'=0.$$

Section II.

The General Equation of the Second Degree.

1. *The general equation of the second degree is*

$$\phi(x, y, z)\equiv ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0, \text{ or } \phi=0.$$

If ρ be the distance between two points (x_1, y_1, z_1), (x_0, y_0, z_0) on the line $\Sigma lx=0$, then (§ 3)

$$\frac{x_1-x_0}{m-n}=\ldots=\ldots=\rho/t.$$

Hence if (x_1, y_1, z_1) lie on $\phi=0$, we have $\phi[(x_0+\overline{m-n}\rho/t), \ldots, \ldots]=0$, or the quadratic (in ρ)

$$\phi(x_0, y_0, z_0)+(\rho/t)\Sigma(m-n)\phi_{x_0}+(\rho^2/t^2)\phi(m-n, n-l, l-m)=0, \ldots\ldots(A)$$

where $$\phi_{x_0}\equiv\frac{\partial\phi}{dx_0}, \text{ etc.}$$

2. *The Tangent.*

If (x_0, y_0, z_0) be on the conic, then $\phi(x_0, y_0, z_0)=0$, and *one* root of (A) is zero for all values of l, m, n. If we choose l, m, n so that $\Sigma(m-n)\phi_{x_0}=0$,

both roots of (A) are zero, *i.e.* the line meets the conic in two coincident points at (x_0, y_0, z_0). If (x, y, z) be any point on this line $\Sigma lx=0$, then

$$\frac{x-x_0}{m-n}=\ldots=\ldots,$$

and the equation of the tangent is $\Sigma(x-x_0)\phi_{x_0}=0$, or (since $\Sigma x_0\phi_{x_0}=0$) $\Sigma x\phi_{x_0}=0$.

3. *The Polar.*

If the tangent at (f, g, h) passes through (x_0, y_0, z_0), $\Sigma x_0\phi_f=0$, and this is identical with $\Sigma f\phi_{x_0}=0$; $\therefore$ (f, g, h) lies on $\Sigma x\phi_{x_0}=0$.

Hence the points of contact of the tangents from (x_0, y_0, z_0) to the conic lie on the straight line $\Sigma x\phi_{x_0}=0$—a real line wherever (x_0, y_0, z_0) may be.

4. *The pair of tangents from a given point.*

In general the roots of (A) are equal (*i.e.* the line $\Sigma lx=0$ is a tangent to $\phi=0$) if $[\Sigma(m-n)\phi_{x_0}]^2=4\phi(x_0y_0z_0)\phi(m-n,\ n-l,\ l=m)$.

Now if (x, y, z) be any point on the line $\Sigma lx=0$, $\dfrac{x-x_0}{m-n}=\ldots=\ldots$, and so the condition becomes

$$[\Sigma x\phi_{x_0}-2\phi(x_0y_0z_0)]^2=4\phi(x_0y_0z_0)\phi(x-x_0,\ y-y_0,\ z-z_0),$$

or

$$[\Sigma x\phi_{x_0}]^2=4\phi(x_0y_0z_0)\phi(xyz).$$

This, then, is the equation of the pair of tangents from (x_0, y_0, z_0).

The tangents to $(uvwu'v'w')(xyz)^2=0$ from the vertex A of the triangle of reference are $Vz^2+Wy^2-2U'yz=0$. Hence the six points of intersection of the tangents from the vertices with the sides of the triangle lie on the conic $\Sigma VWx^2-2\Sigma UU'yz=0$.

Further, since

$$[\Sigma VWx^2-2\Sigma UU'yz]-[\Sigma U'^2x^2-2\Sigma V'W'yz]\equiv H[\Sigma ux^2+2\Sigma u'yz]$$

these two conics and $\Sigma U'x^2-2\Sigma V'W'yz=0$ have a common inscribed quadrilateral. [*Queens' Cam.*, 1898 ; Smith's *Conic Sections*, p. 314, No. 50.]

5. *The Centre.*

If (x_0, y_0, z_0) be chosen so that $\phi_{x_0}=\phi_{y_0}=\phi_{z_0}$ (1) [which with $\Sigma x_0=1$ determine (x_0, y_0, z_0)], then the sum of the roots of (A) is zero, *i.e.* all chords through (x_0, y_0, z_0) are bisected at that point.

Hence (1) gives the centre $(\bar{x}, \bar{y}, \bar{z})$.

Now $$2\phi(\bar{x}, \bar{y}, \bar{z})=\Sigma\bar{x}\frac{d\phi}{d\bar{x}}=\Sigma\bar{x}\,.\,\phi_{\bar{x}}.$$

Eliminate $\bar{x}, \bar{y}, \bar{z}$ from

$$u\bar{x}+w'\bar{y}+v'\bar{z}=w'\bar{x}+v\bar{y}+u'\bar{z}=v'\bar{x}+u'\bar{y}+w\bar{z}=\bar{\phi} \text{ and } \Sigma\bar{x}=1,$$

and we get $K\phi+H=0$, where

$$H=\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} \text{ and } K=\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}.$$

Thus $$\frac{\bar{x}}{U+W'+V'}=\frac{\bar{y}}{W'+V+U'}=\frac{\bar{z}}{V'+U'+W}=-\frac{1}{K}.$$

6. *The Asymptotes.*

If (x_0, y_0, z_0) be taken at the centre, (A) becomes

$$Ht^2 - K\rho^2\phi(m-n,\ n-l,\ l-m)=0.$$

Taking l, m, n so that $\phi(m-n, n-l, l-m)=0$, both roots of this equation become infinite, and hence the equation of the asymptotes is

$$\phi(x-\bar{x},\ y-\bar{y},\ z-\bar{z})=0, \text{ or } K\phi+H(\Sigma x)^2=0.$$

This equation may also be obtained as that of the tangents to the curve from its centre.

COROLLARY 1. "All conics similar, similarly situated, and concentric with $\phi=0$ are included in $\phi+C=0$."

COROLLARY 2. *The Rectangular Hyberbola.*

Since the asymptotes are at right angles, the condition for a rectangular hyperbola is $\Sigma a^2u - 2\Sigma bcu' \cos A=0$.

7. *Condition that a given straight line should touch the conic.*

If $\Sigma lx=0$ touch the conic at (f, g, h), we have $\dfrac{uf+w'g+v'h}{l}=\ldots=\ldots$, and $\Sigma lf=0$. Eliminating (f, g, h), the condition is

$$-\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ l, & m, & n, & 0 \end{vmatrix} \equiv \Sigma Ul^2+2\Sigma U'mn=0.$$

COROLLARY. *Condition for the general equation to represent a parabola.*

Putting $l=m=n=1$, since the line at infinity is a tangent to the curve, the condition becomes $K=-[\Sigma U+2\Sigma U']=0$.

Examples.—The general equation of the inconic is of the form $\Sigma\sqrt{lx}=0$, and $\Sigma Lx=0$ is a tangent if $\Sigma\dfrac{l}{L}=0$.

If the conics $\Sigma\sqrt{l\,.\,x}=0$ and $\Sigma ux(y+z-x)=0$ be similar and similarly situated, two of their common points must lie on the line at infinity, and so for some value of λ,

$$\lambda[\Sigma l^2x^2-2\Sigma mnyz]+\Sigma ux(y+z-x)\equiv(\Sigma x)[(\lambda l^2-u)x+\ldots+\ldots],$$

and $\therefore$ $\lambda m^2-v+\lambda n^2-w=-2\lambda mn+v+w$ and two similar conditions, whence

$$\frac{\lambda}{2}=\frac{v+w}{(m+n)^2}=\ldots=\ldots=\frac{\lambda l^2-u}{(l-m)(l-n)}=\ldots=\ldots, \text{ so that } \Sigma\frac{l}{\lambda l^2-u}=0,$$

and the conics therefore touch one another. [*Jun. Math. Schol., Ox.*, 1898.]

The lines $\Sigma lx=0$ and $\Sigma\lambda x=0$ touch the conic $\Sigma\sqrt{x}=0$ if $\Sigma mn=0$ and $\Sigma\mu\nu=0$, and meet on $\Sigma yz=0$ if $\Sigma(n\lambda-l\nu)(l\mu-m\lambda)=0$. From the first two conditions $l\lambda(m\nu-n\mu)=\ldots=\ldots$, and so the third condition becomes $\Sigma l\lambda=0$. Hence the sides of a triangle inscribed in $\Sigma yz=0$ and described about $\Sigma\sqrt{x}=0$ have equations with the same coefficients with the same cyclical arrangement. [*Sen. Math. Schol., Ox.*, 1898.]

The general equation of all conics touching $lx\pm my\pm nz=0$ is $\Sigma ux^2=0$ where $\Sigma\dfrac{l^2}{u}=0$. The locus of the centres of these conics is $\Sigma l^2x=0$ and the mid-points of the diagonals of the quadrilateral must be on this locus.

[*St. John's* (*Cam.*), 1897 ; *Smith*, p. 297.]

Consider the family of curves $\Sigma ux^2+\lambda\Sigma x^2=0$. The curve is a parabola if $\Sigma\frac{1}{u+\lambda}=0$, and since $(\Sigma u)^2>3\Sigma vw$ this condition gives two real values of λ, *i.e.* the family contains two real parabolae. These curves are convex to the vertex A of the triangle of reference, provided the polar of $A(x=0)$ meets them in real points, *i.e.* if $(v+\lambda)y^2+(w+\lambda)z^2=0$ has real roots, *i.e.* if $-\frac{w+\lambda}{v+\lambda}$ is positive. Denoting this quantity by p, the equation for p is $p^2(w-u)+2p(v-w)+(u-v)=0$; hence, etc. [*King's*, 1894.]

8. From (A) it follows at once that the locus of the mid-points of chords parallel to $\Sigma lx=0$ is $\Sigma(m-n)\phi_x=0$, a straight line passing through the centre and the pole of $\Sigma lx=0$.

(*To be continued.*)

REVIEWS AND NOTICES.

Die Partiellen Differential-Gleichungen der Mathematischen Physik. Nach Riemann's Vorlesungen in vierter Auflage neu bearbeitet von Heinrich Weber. Erster Band. Pp. xviii.—506. (Braunschweig: Vieweg und Sohn, 1900.)

This differs so essentially from the last edition as to be practically a new work. But Professor Weber is fully justified in keeping Riemann's name on the title-page, not only as a pious memorial, but also because he has preserved, as far as possible, the substance of Riemann's work, and in making the requisite additions has happily succeeded in maintaining the spirit of his great predecessor. Among the principal new features in this volume may be mentioned the sections on Bessel functions, vectors, and spherical harmonics, together with a whole book (pp. 305-506) on electricity and magnetism. The analysis is extremely elegant, and shows *inter alia* the advantage of a judicious use of the vector notation.

The physicist will turn with most interest to the third book, where he will find an exposition of electrical theory, with applications to electrostatic equilibrium, Maxwell's electrokinetic equations, electrolytic conduction, and various cases of the flow of electricity in conductors (linear systems, plane, circular disc, surface of a ring, solid sphere, thick plate, solid cylinder, etc.). Professor Weber expressly disclaims the intention of writing a treatise on experimental physics, and contents himself with stating such experimental laws as are required for the mathematical theory. This is a course to which English physicists often object, but it has the advantage of leaving room for the complete mathematical solutions of a large number of problems which are of the highest importance to all electricians. It is to be hoped that our students of electricity will not disdainfully cast aside Professor Weber's work as "mere mathematics"; if they do, it will be to their own loss.

The second volume is to deal with conduction of heat, vibrations (including electric vibrations), elasticity, and hydrodynamics. The order of arrangement of subjects may be thought rather curious; certainly hydrodynamics and the conduction of heat present less difficulty

to a student than electricity does, and naturally precede it in a course of lectures. But Professor Weber is not writing for the average student who is beginning the study of physics, and his arrangement is possibly suggested by a consideration of the analysis required.

G. B. MATHEWS.

Theoretische Arithmetik. VON DR. OTTO STOLZ und DR. J. GMEINER. I. Abtheilung. Pp. iv., 98 (Leipzig: Teubner, 1901).

As indicated on the title-page, this is a revised edition of Sections I.-IV. of Dr. Stolz's *Vorlesungen über allgemeine Arithmetik*, a standard work so well known as to require no commendation. It is a very clear and consistent exposition of the theory of rational numbers, starting with the progression of natural numbers, the characteristic properties of which are stated in the form given by Peano. The work, when complete, will form vol. 4 of Teubner's new series of mathematical text-books. It seems best to reserve detailed criticism until the whole book has appeared.

Practical Mathematics. By A. G. CRACKNELL, M.A., B.Sc. Pp. viii.—368 (Longmans).

The Elements of the Differential and Integral Calculus. By J. W. A. YOUNG and C. E. LINEBARGER. Pp. xviii.—410 (Hirschfeld).

An Elementary Treatise on the Calculus for Engineering Students. By J. GRAHAM, B.A., B.E. Second edition. Pp. xii.—276 (Spon).

The Principles of Mechanics. Part I. By F. SLATE. Pp. x.—300 (Macmillan).

Although these works differ widely both in subject and in scope, they alike illustrate the healthy progress that is being made in bringing the applications of mathematics into closer connection with practical needs, and in clearing away from our text-books those accumulations of unreal and fantastic examples which had grown to be the laughing-stock not only of practical men but of all true mathematicians. Great credit for this improved state of affairs is due to Professor Perry, who has shown both by precept and example the possibility of presenting mathematical theory and practice in such a form as to arouse the interest and stimulate the reasoning power of students whose main concern is with physical science, and for whom mathematics will always be an auxiliary, cultivated not for its own sake so much as for its applications.

Professor Perry has recently expounded his views with great force and eloquence in the columns of *Nature* (July 5 and Aug. 2, 1900); most teachers who have taken the trouble to observe the needs and capacity of the average science student will cordially agree with the bulk of Professor Perry's contention. It is the teaching of mathematics in secondary schools which is most urgently in need of reform. Approximate calculation by decimals, the use of the slide rule and of logarithmic tables, the plotting of simple graphs, are all well within the reach of schoolboys, and should replace the useless work on compound interest, cube root, and circulating decimals which at present takes up so much time. Then in geometry more attention should be paid to practical draughtsmanship, the construction of similar figures to scale, and the elements of solid and descriptive geometry. To bring about a real improvement teachers and examiners must collaborate. So long as examiners set questions of certain stock types, so long will boys be prepared for them; and on the other hand an examiner cannot, without notice, set papers of an entirely novel character.

With all diffidence, I venture to make one representation to the most ardent reformers. They sometimes speak as if the one proper aim of a mathematical teacher should be to provide a "practical" course for the budding engineer. With this view I do not at all agree, even as regards the interests of the engineers themselves. For some years I had the pleasure of teaching some electrical engineers, and I fully admit that their interest in their special subject reacted on their mathematics and made them the best *class* of students I ever had. But what I tried to teach them was the main principles of mathematics, including theoretical dynamics and the elements of the differential and integral calculus.

No attempt was made to give them a special course; except for the calculus (and this was merely a time-table arrangement) they attended my ordinary classes. And I am confident that the outline of general mathematical theory which they obtained was of more value to them than any amount of so-called "heuristic" or rule-of-thumb demonstrations.

To return to the works named at the head of this article, Mr. Cracknell is a more competent mathematician than most authors of books of this kind; at the same time he is eminently practical in the proper sense. His chapters on graphical methods and on logarithms are excellent; so is the one on the slide-rule, though a diagram showing the use of the carrier ought to have been inserted. But this is not very important, because after all the use of the slide-rule can only be learnt by practice with the instrument. The weak point of the book is that lack of space has prevented the author from treating some important subjects in sufficient detail; thus the sections on vectors and on descriptive geometry are very incomplete, and at the same time so good as far as they go as to make the reader regret their brevity.

The treatise of Messrs. Young and Linebarger is based upon that of Nernst and Schönflies. The most interesting thing about it is that one of the authors is a physical chemist, and that examples have been freely drawn from physics and physical chemistry. The book is very well printed, and the integral calculus has been properly introduced at an early stage. There is a preliminary chapter on analytical geometry.

Mr. Graham's work, having reached a second edition, has doubtless been found a useful book. It contains some good illustrations suggested by elastic beams and by electrical problems; in other respects it does not present any very novel features.

Professor Slate's *Principles of Mechanics* is a critical presentation of the subject, influenced more or less by the views of Mach. Probably University students will be those most able to profit by it; it is hardly suitable for beginners. As one of several recent works in which an attempt has been made to free the first principles of mechanics from the appearance of a vicious circle, this treatise certainly deserves attention. G. B. MATHEWS.

A First Geometry Book. By J. G. HAMILTON, B.A., and F. KETTLE, B.A. 1900. (E. Arnold.)

Inductive Geometry for Transition Classes. By H. A. NESBITT. (Sonnenschein.)

Geometrical Drawing for Army and Navy Candidates and Public School Classes. By E. C. PLANT, C.B.

Geometrical Drawing. By W. H. BLYTHE, M.A. Parts I. and II. 1900. (Cam. Univ. Press.)

Exercises in Graphic Statics. By G. F. CHARNOCK, Assoc. M.Inst.C.E. Part I. 40 sheets. 1900. (J. Halden & Co.)

The opinion is gaining ground among teachers that the method of teaching Geometry adopted in this country may have to be considerably altered. Until quite recently Euclid's *Elements* have held their place as the basis, and though attempts have been made to improve geometrical teaching, the reign of Euclid still continues.

The importance of Geometry in a School Curriculum is due in the first place to its utility, and in the second place to the mental training furnished by a strict demonstration of geometrical truths.

Corresponding to this double purpose there should be a double method of teaching, the one part being in each case the complement of the other. Euclid's treatment is essentially Deductive, and should be accompanied and preceded by an Inductive method of teaching, which leads the pupil to discover geometrical truths for himself.

Is it not true that "in this country too much valuable time is spent in the discussion of neat mathematical unrealities?" How few mathematical students really learn to apply their mathematical knowledge to practical use! The reason is to be found in the fact that the practical applications tend to materialise the subject and to bring it down from the region of the eternal. The danger is more

imaginary than real, but it has prevented many engineers and others from acquiring an adequate knowledge of mathematics and the advantages of a mathematical training. Hence the cry for "Practical Mathematics" at the present time.

Even examining bodies are inclined to move out of the beaten path. In the Army Entrance Examinations Euclid and Mensuration are taken together, and some numerical riders are given to the propositions of Euclid. This is surely a step in the right direction.

Messrs. Hamilton and Kettle guide the young pupil by easy stages, and by continually exercising his intelligence, to a practical familiarity with most of the fundamental geometrical truths. There are few teachers who would not profit by reading through this admirable little book. The arrangement is very good and the exercises well chosen. The pupil, however, should be led to see that a figure, though accurately drawn, can only form an approximation to the mental picture intended.

Mr. Nesbitt has written on much the same lines for children of 8 to 12 years of age. He introduces the pupil to the properties of triangles and circles. Mr. Nesbitt appeals more to that intuitive knowledge which enables a child to see the properties of a geometrical figure with his own eyes. There are fewer mechanical exercises, but a good selection of geometrical examples. There is a capital little chapter on Loci, and another on Heights and Distances treated graphically. The proof that the exterior angle of a triangle is greater than each of the interior opposite angles would seem to be a rather hard nut to crack for a child of eight.

Geometrical Drawing is of great utility, and, if taught in direct connection with pure geometry, serves the purpose of illustrating the exact science, and by appealing to the eye assists in familiarising the student with geometrical facts. In too many cases Euclid and Geometrical Drawing are taught independently, where the two subjects should go hand in hand. Will it never be recognised that no real grasp or certainty of handling can be attained without a mastery of principles?

Mr. Plant's text-book on Geometrical Drawing entitled *Practical Plane Geometry*, is beautifully printed on pages of large size, and is designed to present in its diagrams examples of finished mechanical draughtsmanship. The subject-matter is ingeniously cut up into 26 groups named according to the letters of the alphabet, an arrangement which enables the author to throw all natural sequence to the wind, and occasions a needless amount of repetition. Of the five problems in Group I, three have already been worked in Group C, and one other quoted in a preceding group. The problem *To draw a tangent to a given circle* is worked in Group R, but quoted in a preceding Group O. It would be easy to multiply examples.

There are many errors in the letterpress, and frequently confusion arises through two different points in a figure being denoted by the same letter, or through a letter being misplaced.

Here are some specimens of Mr. Plant's "exactness":

"Find EF=side of square=required area, viz. $\sqrt{3''}$."

"A line that expresses the *linear* value of x, the required area."

"Vide Memo on Locii."

"PROPERTIES OF THE ELLIPSE. The ellipse has two focii, and the curve is symmetrical about each of two lines, called the *major*, or conjugate, axis, and the *minor*, or transverse, axis. . . . The focii are obtained by striking an arc from one extremity C of the minor axis, with half the major axis as a radius, to cut the major axis as at F^1, F^2."

Here is an interesting piece of Geometry. "*To bisect any triangle by a line perpendicular to one side.* The reader can supply the figure, taking care to draw the triangle ABC, as Mr. Plant does, with BC longer than AC.

Bisect AB in E. Draw CD perp. to AB. Find AG a mean proportional to AD, AE, and make $AH=AG$. At H draw HK perp. to AB. HK will divide the triangle ABC into two equal areas." Mr. Plant is recommended to work this problem again taking the angle BAC nearly a right angle. It would also be well for him to revise his statement, "The remaining regular polygons having an *odd* number of sides are not capable of containing an inscribed square."

Mr. Blythe's book consists of two parts, of which the second deals with the interesting and important subject of *Descriptive Geometry*.

The arrangement of this work is much better, the problems being grouped according to the geometrical principles upon which their solution depends, but the design is badly carried out. Each chapter commences with a string of geometrical facts, often put together in a very surprising manner and stated with much ambiguity and lack of precision. The frequent references to Euclid seem to suggest that the geometrical truths are based upon the *Elements*. Here is a specimen: "Euclid's propositions about proportion of lines are deduced from the fact that if two triangles are equiangular the sides about the equal angles are proportional." The author then proceeds to prove Euclid VI. 2.

The reader will find that many of the explanations given are very obscure. For instance: "It is necessary to distinguish between linear feet and square feet, thus twice three linear feet is a length of 6 feet, but 2 feet multiplied by 3 feet gives an area of 6 square feet." Here is another example: After stating that a square described on eight-ninths of the diameter of a circle is approximately equal in area to the circle, but that the side so found is really too large by about one-thirtieth of one of the divisions, Mr. Blythe puts the following explanation within brackets:

"To find more exactly the error multiply 81 by 3·141592 and extract the square root, the result is 7·9..." Apart from the numerical mistake, the explanation would appear to suggest that the "one-thirtieth" above is wrong. It would be easy to multiply examples of the lack of mathematical exactness.

We find the term "spherical triangle" applied to a plane figure bounded by three circular arcs, and learn that the normal at any point of the spiral of Archimedes touches a fixed circle. Whenever a geometrical construction is only approximately correct, it would surely be better to say so. For instance, in two consecutive articles a trefoil of semicircular arcs is inscribed in an equilateral triangle, and a quatrefoil in a square. The first construction is exact and the second approximate, but no mention is made of this fact. The second part of the work, as the first, has the same characteristic faults. There is the same apparent effort to put the whole on a satisfactory basis by references to Euclid.

Great prominence is given to the simple proposition: *Given the projections* ab *and* a′ b′ *of a straight line* AB *upon the* H.P. *and* V.P. *respectively, to find its true length and the angle it makes with the* H.P.

The solution of the general case is so simple that there seems to be no advantage in postponing it until particular cases have been discussed. In its actual position the point A is at a distance equal to $a'a_1$ vertically above a, and the point B at a distance equal to $b'b_1$ vertically over the point b. Hence, if we draw aA and bB perpendicular to ab and equal to a_1a' and b_1b' respectively, the figure $abBA$ has only to be turned about ab through a right angle to give the line AB in its true position. Thus the above construction gives the true length of the line, and the angle between ab and AB is the inclination of AB to the $H.P.$; also ab and AB intersect in the $H.T.$ of the line. This is much too easy and straightforward for Mr. Blythe, who considers it necessary before attacking the general case to prove that "when two straight lines are equal and parallel their plans and elevations are equal and parallel." This he does by placing the two lines in a convenient position in a figure and remarking that the proposition is evident from the construction.

Pure Geometry is usually taken as one of the chief subjects in a mathematical training, and rightly so, but in too many cases the student abandons the direct use of geometrical methods in favour of the analytical formulæ to which they give rise. This always occasions a loss of power, while a more frequent return to fundamental geometrical principles is likely to cultivate greater resourcefulness. Consider, for instance, the subject of *Elementary Statics*. How rarely does a pupil learn the practical use of the Triangle of Forces and its converse! Yet he has here a most powerful weapon which he ought to use. It seems a pity that Theoretical Mechanics is in schoolwork divorced from Applied Mechanics. The distinction is still much more real than it should be. At a very early stage, perhaps, in the study of Elementary Statics the pupil should consider the equilibrium of a rod under the influence of forces applied at its extremities only, and then of simple frames loaded at the joints. Then he may consider approximately the equilibrium of simple roof trusses by replacing them by the ideal frames to which they most closely correspond. In this way, though he is conscious that the importance of errors arising from the substitution of

an ideal for the actual structure must be carefully considered at a later date, he gets the full value of a mathematical training and, in addition, profits by its utility.

Mr. Charnock's contribution to Graphic Statics is intended for the use of engineers and others engaged in the design of constructional ironwork. It consists of 40 sheets of diagrams with a brief explanation printed on each sheet, and deals mainly with roof trusses under dead loads and wind pressure. The exercises are taken from actual practice, and some of the stress diagrams are those corresponding to the roofs of well-known railway stations.

The author says that he attempts "to build up the subject from the most elementary principles, the various propositions being proved by the aid of pure geometry." The attempt is highly unsatisfactory, and could give nothing but a confused notion of the fundamental principles of Elementary Statics to any reader who derived his information from no other source.

The Parallelogram of Forces is hopelessly mixed up with the conditions of equilibrium of three forces, no distinction being made between the resultant of two forces and their equilibrant. The author is not careful to give in the right order the letters which name a line representing a force. Thus he says "AC represents a force acting towards A"; and in explaining the Polygon of Forces, we read "It is necessary to join the end of the last line I to the starting point A, and the direction and magnitude of the resultant IA is obtained."

Only a very few elementary propositions are needed, and these could be easily built up within the compass of a few pages. The conditions necessary and sufficient for the equilibrium of two forces acting upon a rigid body having been clearly stated and accepted as axiomatic, the Parallelogram of Forces should be carefully enunciated, and in a work of this kind perhaps assumed and illustrated.

The Parallelogram of Forces is equivalent to the following method for determining the resultant of two forces which act along intersecting lines:

Let two given forces P and Q act along lines which intersect at O. With any suitable scale, draw AB equal to P units of length parallel to the direction of the force P, and from B draw BC equal to Q units of length parallel to the direction of the force Q. Then the straight line from A, where we started, to C, where we finished, represents the resultant in magnitude and direction. Suppose that AC contains R units of length; then the resultant of the two given forces P and Q is a force R acting along a line drawn through O parallel to AC.

We can now proceed to the consideration of the equilibrium of three forces, of any system of forces acting at a point, and of any system of coplanar forces, including parallel forces. With this must be included an explanation and proof of the principle of moments, and "the methods of Ritter, Culmann, and Clerk-Maxwell, are reached."

W. J. DOBBS.

Vocabulaire Mathématique. Part I. (Francais-Allemand). By FELIX MÜLLER. (Pp. xii.—132). 8 marks. 1900 (Teubner).

This is the first part of a dictionary which should be invaluable to students whose knowledge of the German equivalents of technical terms is as yet in its elementary stage—provided of course that they have some knowledge of French. An idea of the richness of the vocabulary will be given by the fact that there are 119 entries under *angle*, 89 under *axis*, 130 under *circle*, 97 under *co-ordinates*, 363 under *curve*, 189 under *equation*, 220 under *function*, 242 under *surface*, and so on. We have noticed one clerical error on p. 16 (*puissane* for *puissance*). Two entries are appended. HOMOLOGIE (une transformation homographique de figures) (Poncelet) [géom. proj.] *Homologie, collineare Verwandtschaft*—DANS L'ESPACE, *collineare Raumverwandtschaft*—INVOLUTIVE, *involutorisch collineare Verwandtschaft*,—DANS LE PLAN, *collineare Verwandtschaft ebener Gebilde*,—DES POLYÈDRES, *collineare Polyederverwandtschaft.*

INFINI, s. (dans la géométrie, Kepler 1615, dans le calcul Fermat, c. 1630) [phil.; fonct. gen.] *Unendliches.*—ABSOLU [fonct. gén.] absolutes Unendlich.—*actuel* (G. Cantor 1883 [phil.; ar. sup.; séries] *aktuelles Unendlich, kategorematisches Unendlich, eigentliches Unendlich.*—D'UNE FONCTION [fonct. gén.] *Unendlich einer Funktion.*—DE L'ORDRE m [ar. sup.] *Unendliches m^{ter} Ordnung.*—POTENTIEL [phil.; ar. sup.; séries] *potentiales Unendlich, synkategorematisches Unendlich, uneigentliches Unendlich.*

Récréations Arithmétiques. By E. Fourrey. Pp. viii.—261. 1899 (Nony). This volume claims to fill the gap between volumes of a similar character, which, says the compiler, are all too old to be of interest, or too advanced to be within the comprehension of the general reader. We suppose the gap was worth filling, but we must confess to finding little in the book that is not to be discovered elsewhere. On the other hand the contents are well grouped and will, no doubt, be read with avidity by many who have a taste for arithmetical conundrums. One interesting feature is that about a quarter of the book is devoted to magic squares, which Fermat described as "rien de plus beau en l'Arithmétique." The explanations throughout are precise and clear, and the book is beautifully printed. We can cordially recommend it as an inexpensive present (3fr. 50c.) to a boy who knows some French and who is fond of arithmetic. It can do him no harm, it will give him much amusement, and it may ripen a latent taste for speculations in a realm which can be explored with fruitful results by a student of ordinary intelligence.

Esercizi ed applicazioni di Trigonometria Piana. By Prof. C. Alasia. Pp. 291. 1901 (Hoepli, Milano). This little volume, as its name implies, consists of exercises on the applications of Plane Trigonometry. The solutions which appear in the text are neat and clear. Limiting values and maxima and minima receive more attention than is usual in elementary works.

Prof. Alasia has elsewhere written on the "Recent Geometry of the Triangle," and his knowledge of such investigations has proved of value in constructing examples. Although the text goes no further than solutions of triangles, and polygons circum- and in-scriptible, the miscellaneous examples contain problems on Brocard angles, etc., and even on harmonic quadrilaterals. These problems require but slender analytical power, yet we cannot but think the student would be better employed in exploring the much wider and more fertile field untouched by this little work. It only remains to say it is beautifully printed on thin glazed paper, and appears to be free from typographical errors.

Annuaire pour l'an 1901, pp. 636, 259. 1fr. 50c. 1901. (Gauthier Villars.) The history of the *Annuaire* is not uninteresting. The Bureau de Longitudes, created by the National Convention [loi du 7 messidor, an iii. (June 25th, 1795)] is compelled to publish every year an *Annuaire*, "propre à régler ceux de toute la République." Neither a coup d'état, nor even the great siege of Paris, has stood in the way of the fulfilment of this obligation. That plucky veteran, M. J. Janssen, who left Paris in a balloon during the siege to witness a solar eclipse, is again to the fore with his account of the work done in 1900 in his Observatory on M. Blanc. We notice that all the dates are in civil mean time, from 0 to 24 o'clock, starting from midnight. The *Annuaire* is the cheapest book we know, containing no less than 636 pages of astronomical, physical, chemical, and statistical tables.

The Teaching of Elementary Mathematics. By D. E. Smith. Pp. 12—312. 1900 (The Macmillan Company).

This is an excellent handbook for the young teacher. It is not as ambitious as the "Methodology" reviewed in our last number, nor does it cover the ground of Laisant's "La Mathématique." But it deals simply and effectively with Arithmetic, Algebra, and Geometry, discussing in each case the historical reasons for teaching the subject, the development of its methods, and the systems in vogue at the present moment. "The effort has been throughout, to set forth the subject in a state of progress to which . . . the teacher is to contribute." Ample bibliographical references are given, but Mr. Smith's apology for this step verges on the comic. "At the risk of being accused of going beyond the needs of teachers, the author has suggested the most helpful works in French and German, as well as in English, and has not hesitated to quote from them." Surely, if the teacher is to contribute to the progress of a subject or to methods of teaching it, this apology is unnecessarily abject! It would have been an advantage had Mr. Smith attached the prices to the books mentioned in "The Teacher's Book Shelf." On p. 299 to Lucas might be added, in a second edition, Cahen's *Théorie des Nombres*, and the forthcoming translation of Dedekind (Open Court Publishing Company).

CORRESPONDENCE.

To the Editor of the "Mathematical Gazette."

Dear Sir,—Your note (*Math. Gazette*, Oct. 1900, p. 397) on marking Euclid papers raises another question: *Should a boy have credit for giving the numbers of the propositions he uses?* I think most people will agree that—with two or three exceptions—a few words are better than a number.

Assuming for the moment that *word-references* are to be preferred to *number-references*, why is it so many boys give the latter? I have quite lately been working with boys fresh from many well-known preparatory schools—they almost all give *number-references*; are they taught to give these or do they do so to save themselves trouble? Judging from the awkward *word-references* they at first give, I am afraid they have never been told to give them. The question may be worthy of discussion, for teachers do not seem unanimous about it.

What are the arguments for *number-references*?

(1) Greater exactness of thought.

(2) Greater brevity, and therefore economy of boy's and master's time and thought.

In answer to

(1) I have found *many* cases where a boy gave the number of a proposition, but could not tell me what the proposition proved.

(2) Economy of time and also thought is a good excuse for the lazy—so far as economy of time is concerned, a little thought will enable the average boy to find a very short way of giving a word reference [*e.g.*:—I. 5, $\angle ABC = \angle ACB$ (angles at base of isos. $\triangle$)—I. 15, $\angle AOC =$ vert. opp. $\angle BOD$—I. 16, $\angle ACD >$ int. opp. $\angle BAC$—etc., etc.]. The great object of all teaching should be to make the pupil think (and thus increase his brain power), so that economy of thought is an evil.

Among arguments for a *word-reference* are the following:

(1) It requires a little more thought (there is great danger of becoming too mechanical in this age of bustle and hurry).

(2) It brings the proposition referred to and its essential points more vividly before the mind.

(3) It shows up the boy who has tried to learn his propositions with as little thought as possible and is a safeguard for the thoughtful boy (it is easy to mean the right proposition and give the wrong number).

(4) Great weakness is displayed in geometrical conics by boys who have been in the habit of giving number-references; in writing out propositions or riders, they at first refer to conic properties by number (which is absolutely useless and always will be unless a second Euclid is born), and then they give no references at all (which is perhaps worse).

It would be a great help to teachers generally to hear other people's opinions on this question, and the organ of (what was once) the A.I.G.T. is surely the proper place for such opinions to appear. The question is one of considerable importance and, though I fancy many (if not most) teachers will agree with me, I am afraid many do not practise what they preach.

Of course there are exceptions to every rule; references by number to such propositions as I. 4, I. 8, I. 26, and I. 47, I should certainly allow.—I am, yours truly,

A. W. S.

Harrow School.

MATHEMATICAL NOTES.

95. [**D. 2. d. a.**] *Note on Periodic Continued Fractions.*

1. If ϕ be the periodic continued fraction

$$\dot{a}_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\ldots+\frac{1}{\dot{a}_n+}\,\frac{1}{a_1+}\ldots,$$

we have
$$\phi = a_1 + \frac{1}{a_2+}\dots\frac{1}{+a_n+}\frac{1}{\phi},$$
$$= \frac{(a_1, a_2, \dots, a_n, \phi)}{(a_2, \dots, a_n, \phi)},$$

where $(a_1, a_2, a_3, \dots)$ is the simple continuant whose main diagonal is $a_1, a_2, a_3, \dots$. From this it follows that

$$\phi = \frac{(a_1, a_2, \dots, a_n)\phi + (a_1, a_2, \dots, a_{n-1})}{(a_2, \dots, a_n)\phi + (a_2, \dots, a_{n-1})}$$
$$= \frac{A\phi + A'}{B\phi + B'}, \text{ say };$$

and therefore
$$B\phi^2 + (B' - A)\phi - A' = 0,$$

whence, by reason of the relation $A'B - AB' = (-1)^{n-1}$, we obtain ϕ in terms of A, B, B'. viz.:

$$\phi = \frac{1}{2B}\{A - B' \pm \sqrt{(B' - A)^2 + 4A'B}\}$$
$$= \frac{1}{2B}\{A - B' \pm \sqrt{(A + B')^2 + 4(A'B - AB')}\}$$
$$= \frac{1}{2B}\{A - B' \pm \sqrt{(A + B')^2 + (-1)^{n-1}4}\}.$$

2. The product of the n continued fractions which have the same period as ϕ

$$= \frac{(a_1, a_2, \dots, a_n, \phi)}{(a_2, \dots, a_n, \phi)} \times \frac{(a_2, a_3, \dots, a_n, \phi)}{(a_3, \dots, a_n, \phi)} \times \dots \frac{(a_n, \phi)}{\phi}$$
$$= \frac{(a_1, a_2, \dots, a_n, \phi)}{\phi},$$

and therefore, from what appears in § 1,
$$= (a_2, a_3, \dots, a_n, \phi).$$

3. This result however is equal to $B\phi + B'$; and substituting the value of ϕ above obtained we have the product in question

$$= \tfrac{1}{2}\{A + B' \pm \sqrt{(A + B')^2 + (-1)^{n-1}4}\},$$

that is to say, equal to a function of only one argument, $A + B'$.

4. When the number of elements in the period is odd this function is equal to

$$A + B' + \frac{1}{A + B' +}\frac{1}{A + B' + \dots},$$

the result then taking the form of a generalisation of Mr. Genese's interesting problem, No. 383 (Vol. I., p. 394), which in fact suggested the present note.

The still more general theorem may be formally enunciated as follows:

If ϕ be the periodic continued fraction

$$\dot{a}_1 + \frac{1}{a_2+}\dots\frac{1}{+\dot{a}_n+}\frac{1}{a_1+}\dots,$$

then
$$\frac{(a_1, \dots, a_n, a_1, \dots, a_n, a_1, \dots)}{(a_1, \dots, a_n, a_1, \dots)} = (a_2, a_3, \dots, a_n, \phi),$$
$$= \tfrac{1}{2}\{t \pm \sqrt{t^2 + (-1)^{n-1}4}\},$$

where
$$t = (a_1, a_2, \dots, a_n) + (a_2, a_3, \dots, a_{n-1}).$$

T. MUIR.

96. [A. 1. a.] *Note on the Convergency of the Geometric Series.*

The following proposition is needed to complete the proof of the convergency of a geometric series when the common ratio is a proper fraction.

If r is a positive proper fraction less than 1 by a finite proper fraction, l suppose, r^n may be made less than any assigned quantity by taking n large enough.

This may be proved as follows:

We know that $$r^n+l(1+r+r^2+\dots+r^{n-1})=1.$$
$$\therefore\ r^n+nlr^{n-1}<1;$$
now all the symbols represent positive quantities,
$$\therefore\ r^n<r/(r+nl).$$

By taking n large enough $r+nl$ can be made larger than any assigned quantity; hence by taking n large enough, r^n can be made less than any assigned quantity. W. H. H. Hudson.

PROBLEMS.

[*Much time and trouble will be saved the Editor if (even tentative) solutions are sent with problems by their Proposers.*]

420. [K. 2. b.] (1) The incircle and the excircle opposite A of a triangle ABC touch the sides at D, E, F and D', E', F' respectively and $EF, E'F'$ produced cut BC produced in K, K'; show that $KD=D'K'$.

[K. 16. b. α.] (2) A sphere is described to touch the edges of a tetrahedron; show that, of the 15 lines which join the points of contact, 3 are concurrent and the rest meet in pairs on the edges of the tetrahedron. W. F. Beard.

421. [D. 2. d.] (1) If $\dfrac{p_n}{q_n}$ be the n^{th} convergent of $a-\dfrac{1}{a-}\,\dfrac{1}{a-\dots}$, shew that
$$p_n^2-ap_nq_n+q_n^2=p_n^2-p_{n+1}p_{n-1}=1,$$
and find the value of
$$a^2(p_nq_{n-1}+q_np_{n-1})-2a(p_np_{n-1}+q_nq_{n-1}).$$

(2) If $\dfrac{\pi}{2}, \alpha_1, \alpha_2, \dots \alpha_n, \dots$ be a series of angles such that
$$\cot\alpha_2=\mu\cot\alpha_1-\cot\frac{\pi}{2},$$
and, generally, $\cot\alpha_n=\mu\cot\alpha_{n-1}-\cot\alpha_{n-2}$ where $\mu=2\operatorname{cosec}\alpha_1$

show also that $$\operatorname{cosec}\alpha_2=\mu\operatorname{cosec}\alpha_1-\operatorname{cosec}\frac{\pi}{2},$$

and, generally, $$\operatorname{cosec}\alpha_n=\mu\operatorname{cosec}\alpha_{n-1}-\operatorname{cosec}\alpha_{n-2}.$$ E. Budden.

422. [K. 1. a.] Three collinear points are at distances a_1, a_2, a_3 from one another and r_1, r_2, r_3 from a fourth point distant p from their line; prove that
$$\Sigma a=0,$$
$$\Sigma ar^2+\Pi a=0,$$
$$p^2\Pi a^2=4s\Pi(s-ar),$$
where $$2s=\Sigma ar.$$ R. W. H. T. Hudson.

423. [B. 1. a.] Evaluate in terms of the zeros of ax^2+bx+c, or otherwise, the determinant of the n^{th} order

$$\begin{vmatrix} b+d, & c+d, & d, & d & \dots d \\ a+d, & b+d, & c+d, & d & \dots d \\ d, & a+d, & b+d, & c+d & \dots d \\ d, & d, & a+d, & b+d & \dots d \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

in which each diagonal is composed of the same elements and all the elements outside the central three diagonals are the same.

Show also that the determinant is the coefficient of x^n in the expansion of

$$\frac{1}{1-bx+acx^2}+\frac{dx(1-acx^2)}{(1+ax)(1+cx)(1-bx+acx^2)^2}.$$ F. S. MACAULAY.

424. [R. 1. a.] Two points moving at equal speeds, one on a circle, the other on a tangent, arrive at the point of contact simultaneously. Where does the line joining them meet the diameter through the point of contact *just before* they coincide? C. E. M'VICKER.

425. [K. 2. c.] The sides of a triangle ABC are bisected in D, E, F; show that on the circle DEF four positions of a point P may be found such that $DP \pm EP \pm FP = 0$, and that these are the four points where the circle is touched by the incircle and excircles of the triangle ABC.

Show also that the tangents to the circle at these four points are also tangents of the ellipse which touches each side of ABC at its middle point. H. W. RICHMOND.

426. [L[1]. 16. a.] If the sides of a triangle are $\sqrt{pk \pm a^2}$, $\sqrt{pk \pm b^2}$, $\sqrt{pk \pm c^2}$, where p is any positive integer and $k = \Sigma a^2$; ω_p, e_p, the Brocard angle and eccentricity of the Brocard ellipse; $\lambda^2 = \Sigma a^2 b^2$, $\nu^4 = \Sigma a^4$, prove that

$$\cot \omega_p = (3p \pm 1)k/D; \quad \lambda_p^2 e_p^2 = \lambda^2 e^2,$$

where $$D^2 = (3p^2 \pm 2p)k^2 + 16\Delta^2;$$

also find $\cot A_p$ and $\sum_1^r (e_p^{-2} + e'^{-2}_p)$, where the accent means that the negative sign is taken in the ambiguities. R. TUCKER.

427. [K. 12. b.] Four circles can be drawn to touch two given lines AB, AC, and (at T, T_1, T_2, T_3 respectively) a given circle ABC; prove this construction: bisect the arcs BC at D, D'; with centres D, D' draw circles BC; let II_1 and I_2I_3 be the diameters of these circles which pass through A; then the lines joining each centre to the ends of the other diameter determine T, T_1, T_2, T_3. Show also that TT_1 and T_2T_3 meet BC on the tangent at A to the circle ABC. C. E. YOUNGMAN.

428. [R. 9. b.] Two equal smooth spheres are moving with uniform velocities U and V on a smooth horizontal table; at one instant the distance between their centres is a times the sum of their radii, and their directions of motion make angles θ and ϕ with the line joining their centres at that instant; prove that if the spheres collide the impulse between them will be

$$I\{V^2 + U^2 - 2UV\cos(\theta - \phi) - a^2(V\sin\phi - U\sin\theta)^2\}^{\frac{1}{2}},$$

where I is the impulse which would be produced by direct collision with unit relative velocity. Trin. (C.), 1892.

429. [R. 4. a.] Three equal smooth spheres of radius r rest in a hollow hemisphere of radius R with their centres in the same horizontal plane; a cone whose weight is equal to the weight of a sphere and whose semivertical angle is a is inserted symmetrically between the spheres with its vertex downwards; prove that the spheres will separate if

$$\cot a > \frac{8r}{\sqrt{3R^2 - 6Rr - r^2}}.$$ Pemb. (C.), 1893.

SOLUTIONS.

UNSOLVED QUESTIONS.—171, 275, 279, 283, 326, 336-8, 341, 349, 356, 369, 370, 373-9, 381-2, 389, 393, 395-7, 401, 404, 406.

Solutions to the above, or other questions to which no solution has yet been published, and to 410-19 should be sent as early as possible.

The question need not be re-written; the number should precede the solution. Figures should be **very carefully drawn to a small scale** on a separate sheet.

384. [L_1. 17. e.] *Two conics intersect at right angles at each vertex of a given right-angled triangle. Show that they must be confocals, or if not, find the locus of their remaining point of intersection.* E. INNES.

Solution by W. F. BEARD.

Let AOB be the $\triangle$ with a right $\angle$ at O, and let $OA=k$, $OB=l$.

Let equations of conics referred to OA, OB as axes be

$$ax^2+2hxy+by^2-akx-bly=0\ ;\quad a_1x^2+2h_1xy+b_1y^2-a_1kx-b_1ly=0.$$

Since they intersect orthogonally at O; $\therefore\ aa_1k^2+bb_1l^2=0.$ ....(i)

Tangents at (k, o) are $\quad akx+(2hk-bl)y-ak^2=0$;

$$a_1kx+(2h_1k-b_1l)y-a_1k^2=0.$$

Since these are perpendicular;

$$\therefore\ aa_1k^2+(2hk-bl)(2h_1k-b_1l)=0\ ;$$

from (i.) this becomes $\quad 2hh_1k-l(bh_1+b_1h)=0$ ....(ii)

Similarly for (o, l), $\quad 2hh_1l-k(ah_1+a_1h)=0.$ ....(iii)

Let P be the fourth point of intersection and let equation of OP be

$$mx+ny=0.$$

Then for some value of λ,

$$ax^2+2hxy+by^2\ldots+\lambda(a_1x^2+2h_1xy+\ldots)\equiv(mx+ny)(lx+ky-kl)\ ;$$

$$\therefore\ \frac{2lm}{a+\lambda a_1}=\frac{ln+km}{h+\lambda h_1}=\frac{2kn}{b+\lambda b_1}\ ;$$

$$\therefore\ \lambda\{(ln+km)a_1-2h_1lm\}=-\{(ln+km)a-2hlm\},$$

and $\quad \lambda\{(ln+km)b_1-2h_1kn\}=-\{(ln+km)b-2hkn\}.$

Eliminating λ, we get

$$(ln+km)^2(a_1b-ab_1)-2(ln+km)\{lm(bh_1-b_1h)+kn(a_1h-ah_1)\}=0\ ;$$

$$\therefore\ ln+km=0, \quad\text{.....................................(iv)}$$

or

$$\frac{m}{n}=\frac{2k(a_1h-ah_1)-l(a_1b-ab_1)}{k(a_1b-ab_1)-2l(bh_1-b_1h)}$$

$$=\frac{a_1(2hk-bl)-a(2h_1k-b_1l)}{b_1(2hl-ak)-b(2h_1l-a_1k)}$$

$$=\frac{\dfrac{a_1b_1hl}{h_1}-\dfrac{abh_1l}{h}}{\dfrac{a_1b_1hk}{h_1}-\dfrac{abh_1k}{h}}\quad\text{[by (ii) and (iii)]}$$

$$=\frac{l}{k}.$$

$\therefore$ equation of OP is either $lx+ky=0$ or $lx=ky$, from (iv).

If the equation of OP is $lx=ky$, we have

$$\frac{ak^2+bl^2}{a_1k^2+b_1l^2}=\frac{ak^2+bl^2+2hkl}{a_1k^2+b_1l^2+2h_1kl}$$

and $\quad \therefore\ =\dfrac{h}{h_1}.$

But from (ii) and (iii) $$\frac{h}{h_1} = -\frac{ak^2 - bl^2}{a_1k^2 - b_1l^2};$$

$$\therefore \text{ either } \frac{ak^2+bl^2}{a_1k^2+b_1l^2} = \frac{ak^2-bl^2}{b_1l^2-a_1k^2} = \frac{ak^2}{b_1l^2} = \frac{bl^2}{a_1k^2},$$

which gives $\frac{aa_1}{bb_1} = \frac{l^4}{k^4}$, which is inconsistent with (i), or $h = h_1 = 0$, in which case (k, l) is on both conics and it readily follows that the conics are confocal.

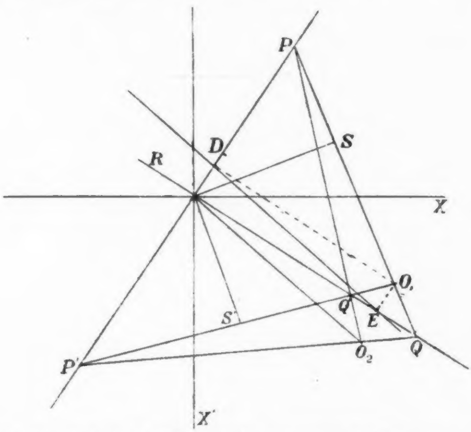
$\therefore$ Unless the conics are confocal the locus of P is the straight line through O parallel to AB, *i.e.* $$lx + ky = 0.$$

Or by reciprocation and geometry, thus:

Reciprocate with regard to O, and the theorem becomes:

Given two parabolas with their axes at right angles touching two fixed perpendicular straight lines, such that the parts of these tangents intercepted between the parabolas subtend right angles at a fixed point, find the *envelope* of the third common tangent.

Since the two given common tangents are perpendicular, they intersect at the point of intersection of the directrices.



Let S, S' be the foci, PP', QQ' the given perpendicular common tangents.

Let PP', QQ' meet at R. Draw RX, RX' parallel to the axes.

Produce $P'Q'$, PQ' to meet PQ, $P'Q$ at O_1, O_2.

Then

$$O_1\hat{Q}'Q = R\hat{Q}'P' = X'\hat{R}Q' \; [\because \; P'Q' \text{ passes through } S'; \; \therefore \; X'\hat{R}Q' = S'\hat{Q}'R.]$$
$$= X\hat{R}P; \quad \because \; X'\hat{R}X = Q\hat{R}P$$
$$= S\hat{P}R \; [\because \; PQ \text{ goes through } S; \; \therefore \; X\hat{R}P = S\hat{P}R.];$$
$$\therefore \; RQ'O_1P \text{ are concyclic, III. 22,}$$
$$\text{and } \therefore \; Q\hat{O}_1Q' = P\hat{R}Q' = \text{right angle};$$
$$\therefore \; Q' \text{ is orthocentre of } PP'Q, \text{ and } \therefore \; PQ' \text{ is perpendicular to } P'Q.$$

Thus the circles on PP', QQ' as diameters meet at O_1O_2, and $\therefore$ O_1, O_2 are the only points at which PP', QQ' subtend right angles.

Thus either O_1 or O_2 is the fixed point given, viz., O in original problem.

Let the third common tangent to the parabolas meet PP', QQ' in D, E.

Then circle round RDE goes through S, S', but $S\hat{R}S'$ = right angle;

$\therefore$ D, E are the points in which the circle on SS', or on RO_1 as diameter meets PP', QQ';

$$\therefore\ O_1D,\ O_1E \text{ are perpendicular to } PP',\ QQ'.$$

(i) Thus if O_1 is given fixed point, DE is a fixed straight line.

(ii) If O_2 is the given fixed point.

$$\begin{aligned} \text{Then } D\hat{E}R &= E\hat{R}O_1\,; \because RO_1 \text{ is a rectangle} \\ &= E\hat{R}O_2\,; \because RO_2O_1 \text{ is pedal triangle of } PP'Q \\ &\therefore DE \text{ is parallel to } RO_2. \end{aligned}$$

Thus the third common tangent is either fixed in (i), or is parallel to a fixed straight line—that joining O to the intersection of the fixed lines—in (ii).

Hence in our original theorem, the fourth point of intersection P is either a fixed point or on a fixed straight line through the point O, and parallel to that joining A and B.

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* Will be reviewed shortly.

ERRATUM.

Delete solution to 366 [K. 3. b.] p. 42. A correct solution will be found on p. 21.

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